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THE BOUSSINESQ PROBLEM FOR SOILS WITH BOUNDED NON-HOMOGENEITY

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SUMMARY

The response of a compressible continuously non-homogeneous elastic soil to a static vertical point load on its surface is analytically investigated by using classical integral transform techniques and the extended power series method for obtaining the solution in the transform domain. The non-homogeneity is described by means of a depth-function which is non-zero at the surface and bounded at infinity and is capable in modelling both increasing and decreasing soil stiffness with depth. The influence of non-homogeneity on the displacements and stresses at the surface and in the interior is examined over a wide range on the governing parameters. © 1998 John Wiley & Sons, Ltd.

Key words: elastic nonhomogeneity; soil deformation; soil stresses; surface loading

INTRODUCTION

Although a variety of sophisticated constitutive laws have been developed during the last decades to describe anelastic soil behaviour, solutions based on linear elastic theory are still a convenient and rational tool in geotechnical practice provided that the level of induced stresses in the soil is considerably lower than its ultimate strength and appropriate equivalent elastic parameters are selected for the soil. Applications include deformations of geotechnical structures under working loads, interpretation of *in situ* tests or selection of stress paths to guide appropriate laboratory testing. Finally, analytical solutions of elasticity are well-suited as benchmarks for computer codes.

Boussinesq's[1] solution for the response of a linear-elastic homogeneous half-space to a concentrated vertical load on its surface provides the basis for estimating stresses and displacements within a soil mass. However, even in non-layered soils increasing overburden pressure causes stiffness to become greater with depth. Analytical continuum models reflecting this property of uniformly deposited soils were first studied by Gibson[2] by assuming incompressibility and a linear increase of soil stiffness with depth. In later papers Gibson and co-workers treated problems in which the non-homogeneous medium has a Poisson's ratio other than one-half[3,4] or/and is of finite depth.[5,6] The normal surface loading of an incompressible linearly non-homogeneous half-space has been re-examined by Calladine and Greenwood.[7] The interior

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loading of such a half-space has been studied by Rajapakse.[8] Power laws for the depth variation of soil stiffness have also been considered. Except the work by Booker *et al.*,[9] a restriction linking the exponent of the power law to the Poisson's ratio was used.[10,11]

The inherent deficiency of these models with respect to geotechnical applications is the vanishing soil stiffness at the surface and/or its unboundedness at infinite depth. Avoiding such constraints, Selvadurai *et al.*[12] introduced a non-homogeneous soil model in which the stiffness varies in an exponential fashion with depth being bounded at infinity and solved the torsional indentation problem for a rigid circular punch. Very recently Selvadurai[13] treated the corresponding axisymmetric contact problem by numerically solving the associated integral equation. Adopting the same soil model, the author[14] recently presented an analytical solution for the dynamic counterpart of Boussinesq's problem.

Discrete methods for multilayered soils[15–17] may also be used in the analysis of non-homogeneous soils by replacing the continuous profile by an assembly of elastic layers resting on a homogeneous half-space.

In the sequel, stresses and displacements of a non-homogeneous compressible elastic soil subjected to a surface static point load are determined. Stiffness increase or decrease with depth is described by means of a bounded depth-function of the shear modulus. Hankel transform techniques are employed leading to a system of ordinary differential equations in the transform domain which is solved analytically. Selected numerical results are presented demonstrating the influence of non-homogeneity and Poisson's ratio on soil response.

GOVERNING EQUATIONS AND GENERAL SOLUTION

We consider the axisymmetric problem of a linear-elastic, isotropic half-space occupying the region $z \geqslant 0$ loaded by a static load Q on the surface, as shown in Figure 1. The half-space is described by a constant Poisson's ratio $0 \leqslant \nu < 0{\cdot}5$ and shear modulus G varying with depth such that

$$G(z) = G_0 + (G_\infty - G_0)(1 - \mathrm{e}^{-\alpha z}) \tag{1}$$

G_0 and G_∞ are the shear moduli at the surface and at infinite depth, respectively, and α is a constant which is referred to as gradient of non-homogeneity. The derivation follows the same lines as for the solution of the dynamic problem in Reference 14 and for the sake of conciseness only the essential equations are given in the following.

Combining the equations of static equilibrium, the strain–displacement relations and the constitutive equation for linear-elastic material the following system of coupled partial differential equations is obtained for the displacements $u(r, z)$ and $w(r, z)$ in the r- and z-directions, respectively:

$$\frac{2G(1-\nu)}{(1-2\nu)}\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} - \frac{u}{r^2}\right) + G\frac{\partial^2 u}{\partial z^2} + \frac{G}{1-2\nu}\frac{\partial^2 w}{\partial r \partial z} + \frac{\mathrm{d}G}{\mathrm{d}z}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}\right) = 0 \tag{2}$$

$$G\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r}\right) + \frac{2(1-\nu)}{1-2\nu}\left(G\frac{\partial^2 w}{\partial z^2} + \frac{\mathrm{d}G}{\mathrm{d}z}\frac{\partial w}{\partial z}\right) + \frac{2\nu}{1-2\nu}\frac{\mathrm{d}G}{\mathrm{d}z}\left(\frac{\partial u}{\partial r} + \frac{u}{r}\right)$$
$$+ \frac{G}{1-2\nu}\left(\frac{1}{r}\frac{\partial u}{\partial z} + \frac{\partial^2 u}{\partial r \partial z}\right) = 0 \tag{3}$$

where $G = G(z)$ is defined by equation (1).

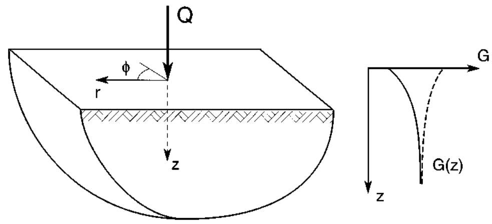

Figure 1. The problem under consideration

The associated stresses are obtained from the displacements by

$$\sigma_{rr} = \frac{2G\nu}{1-2\nu}\left(\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{\partial w}{\partial z}\right) + 2G\frac{\partial u}{\partial r} \tag{4}$$

$$\sigma_{zz} = \frac{2G\nu}{1-2\nu}\left(\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{\partial w}{\partial z}\right) + 2G\frac{\partial w}{\partial z} \tag{5}$$

$$\sigma_{\phi\phi} = \frac{2G\nu}{1-2\nu}\left(\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{\partial w}{\partial z}\right) + 2G\frac{u}{r} \tag{6}$$

$$\sigma_{rz} = \sigma_{zr} = G\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}\right) \tag{7}$$

The sign convention of the elasticity theory is used, i.e. tensile stress is positive.

The boundary conditions require bounded stresses and displacements at remote distances and on the half-space surface

$$\text{for} \quad z = 0 \qquad \sigma_{zz}(r, z) = -Q\delta(r) \tag{8}$$

$$\sigma_{rz}(r, z) = 0 \tag{9}$$

where $\delta(\cdot)$ is the Dirac delta function.

Employing a Hankel transform over the radial co-ordinate r

$$u(r, z) = \int_0^\infty k\bar{u}(k, z)J_1(kr)\,\mathrm{d}k \tag{10}$$

$$w(r, z) = \int_0^\infty k\bar{w}(k, z)J_0(kr)\,\mathrm{d}k \tag{11}$$

where k is the Hankel transform parameter and J_n is the Bessel function of the first kind of order n, we obtain from equations (2) and (3) the following set of ordinary differential equations:

$$G\frac{\partial^2\bar{u}}{\partial z^2} + \frac{\mathrm{d}G}{\mathrm{d}z}\frac{\partial\bar{u}}{\partial z} - \frac{k^2G}{\bar{\nu}}\bar{u} - \frac{1-\bar{\nu}}{\bar{\nu}}kG\frac{\partial\bar{w}}{\partial z} - k\frac{\mathrm{d}G}{\mathrm{d}z}\bar{w} = 0 \tag{12}$$

$$G\frac{\partial^2\bar{w}}{\partial z^2} + \frac{\mathrm{d}G}{\mathrm{d}z}\frac{\partial\bar{w}}{\partial z} - \bar{\nu}k^2G\bar{w} + (1-\bar{\nu})kG\frac{\partial\bar{u}}{\partial z} + (1-2\bar{\nu})\,k\frac{\mathrm{d}G}{\mathrm{d}z}\bar{u} = 0 \tag{13}$$

where

$$\bar{v} = \frac{1 - 2v}{2(1 - v)} \tag{14}$$

By means of an appropriate transformation of the depth co-ordinate the unbounded domain $0 \leqslant z < \infty$ can be mapped onto a bounded one. We will distinguish between two cases describing increasing and decreasing shear modulus with depth, respectively:

(i) $G_0 \leqslant G_\infty$:

Introducing

$$\xi = \Xi_0 \, \mathrm{e}^{-\alpha z} \tag{15}$$

where

$$\Xi_0 = 1 - \frac{G_0}{G_\infty} \tag{16}$$

the shear modulus variation is given by

$$G = G_\infty (1 - \xi) \tag{17}$$

and the system of differential equations (12) and (13) is transformed to

$$\alpha^2 \bar{v} \xi^2 (1 - \xi) \bar{u}'' + \alpha^2 \bar{v} \xi (1 - 2\xi) \bar{u}' - k^2 (1 - \xi) \bar{u} + \alpha (1 - \bar{v}) k \xi (1 - \xi) \bar{w}' - \alpha \bar{v} k \xi \bar{w} = 0 \tag{18}$$

and

$$\begin{aligned} \alpha^2 \xi^2 (1 - \xi) \bar{w}'' + \alpha^2 \xi (1 - 2\xi) \bar{w}' - \bar{v} k^2 (1 - \xi) \bar{w} \\ - \alpha (1 - \bar{v}) k \xi (1 - \xi) \bar{u}' + \alpha (1 - 2\bar{v}) k \xi \bar{u} = 0 \end{aligned} \tag{19}$$

where ( )$'$ denotes differentiation with respect to ξ.

(ii) $G_0 \geqslant G_\infty$:

In analogy to case (i), we introduce

$$\xi = \Xi_0^* \exp(-\alpha z) \tag{20}$$

with

$$\Xi_0^* = 1 - \frac{G_\infty}{G_0} \tag{21}$$

yielding

$$G = G_0 (1 - \Xi_0^* + \xi) \tag{22}$$

The differential equations (12) and (13) are than transformed to

$$\begin{aligned} \alpha^2 \bar{v} \xi^2 (1 - \Xi_0^* + \xi) \bar{u}'' + \alpha^2 \bar{v} \xi (1 - \Xi_0^* + 2\xi) \bar{u}' - k^2 (1 - \Xi_0^* + \xi) \bar{u} \\ + \alpha (1 - \bar{v}) k \xi (1 - \Xi_0^* + \xi) \bar{w}' + \alpha \bar{v} k \xi \bar{w} = 0 \end{aligned} \tag{23}$$

and

$$\alpha^2\xi^2(1-\Xi_0^*+\xi)\bar{w}''+\alpha^2\xi(1-\Xi_0^*+2\xi)\bar{w}'-\bar{v}k^2(1-\Xi_0^*+\xi)\bar{w}$$
$$-\alpha(1-\bar{v})k\xi(1-\Xi_0^*+\xi)\bar{u}'-\alpha(1-2\bar{v})k\xi\bar{u}=0 \tag{24}$$

Ξ_0 and Ξ_0^* are equivalent to each other measures of the non-homogeneity of the half-space medium and are referred to as degree of non-homogeneity. The homogeneous half-space is recovered either by setting $\Xi_0=0$ or by the limiting case $\alpha\to 0$.

We first consider the case (i). The extended power series method is applied to find analytical solutions for the system of linear differential equations (18) and (19). The procedure for the dynamic case is outlined in Reference 18. A slightly different treatment is required for the static case considered herein.

The general solution has the form

$$\bar{u}(k,\xi)=\sum_{i=1}^{4}A_i(k)\bar{u}^{(i)}(k,\xi) \tag{25}$$

$$\bar{w}(k,\xi)=\sum_{i=1}^{4}A_i(k)\bar{w}^{(i)}(k,\xi) \tag{26}$$

the functions $\bar{u}^{(i)}$ and $\bar{w}^{(i)}$ being expressed in terms of power series. Substituting

$$\bar{u}^{(i)}(k,\xi)=\sum_{n=0}^{\infty}a_n\xi^{n+l} \tag{27}$$

$$\bar{w}^{(i)}(k,\xi)=\sum_{n=0}^{\infty}b_n\xi^{n+l} \tag{28}$$

into differential equations (18) and (19) we obtain the indicial equation for l which has two double roots

$$l_{1/2}=m \qquad l_{3/4}=-m \tag{29}$$

where

$$m=\frac{k}{\alpha} \tag{30}$$

This indicates that one set of solutions will be of the form (27)–(28) while the second one must contain a logarithmic term.[19] In order to satisfy the boundedness condition for large z (i.e. small ξ) the third and fourth root are rejected by setting

$$A_3=A_4=0 \tag{31}$$

Thus, the first solution reads

$$\bar{u}^{(1)}(k,\xi)=\sum_{n=0}^{\infty}a_n^{(1)}\xi^{n+m} \tag{32}$$

$$\bar{w}^{(1)}(k,\xi)=\sum_{n=0}^{\infty}b_n^{(1)}\xi^{n+m} \tag{33}$$

The coefficients $a_n^{(1)}$ and $b_n^{(1)}$ are determined for

$$a_0^{(1)} = 1 \qquad b_0^{(1)} = 1 \tag{34}$$

and $i = 1$ by the two coupled recurrence relations

$$a_n^{(i)} = \frac{\bar{E}_1^{(i)} E_{22} - \bar{E}_2^{(i)} E_{12}}{E_D} \tag{35}$$

$$b_n^{(i)} = \frac{\bar{E}_2^{(i)} E_{11} - \bar{E}_1^{(i)} E_{12}}{E_D} \tag{36}$$

with

$$\bar{E}_1^{(1)} = \bar{E}_{11} a_{n-1}^{(1)} + \bar{E}_{12} b_{n-1}^{(1)} \tag{37}$$

$$\bar{E}_2^{(1)} = \bar{E}_{21} a_{n-1}^{(1)} + \bar{E}_{22} b_{n-1}^{(1)} \tag{38}$$

$$E_D = -\, n^2 (n + 2m)^2 \bar{v} \tag{39}$$

where

$$E_{11} = \bar{v}(n + m)^2 - m^2 \tag{40}$$

$$\bar{E}_{12} = m(1 - \bar{v})(n + m) \tag{41}$$

$$E_{22} = \bar{v} m^2 - (n + m)^2 \tag{42}$$

$$\bar{E}_{11} = \bar{v}(n + m)(n + m - 1) - m^2 \tag{43}$$

$$\bar{E}_{12} = m[(n + m - 1) - \bar{v}(n + m - 2)] \tag{44}$$

$$\bar{E}_{21} = m[(1 - \bar{v})(n + m) - \bar{v}] \tag{45}$$

$$\bar{E}_{22} = \bar{v} m^2 - (n + m)(n + m - 1) \tag{46}$$

The second solution has the form[19]

$$\bar{u}^{(2)}(k, \xi) = \bar{u}^{(1)} \ln \xi + \sum_{n=0}^{\infty} a_n^{(2)} \xi^{n+m} \tag{47}$$

$$\bar{w}^{(2)}(k, \xi) = \bar{w}^{(1)} \ln \xi + \sum_{n=0}^{\infty} b_n^{(2)} \xi^{n+m} \tag{48}$$

the power series coefficients being determined for

$$a_0^{(2)} = 1 \tag{49}$$

by

$$b_0^{(2)} = 1 - \frac{1 + \bar{v}}{m(1 - \bar{v})} \tag{50}$$

and for $n \geqslant 1$ by the coupled recurrence relations (35) and (36) with $i = 2$ and

$$\begin{aligned} \bar{E}_1^{(2)} = {} & \bar{E}_{11} a_{n-1}^{(2)} + \bar{E}_{12} b_{n-1}^{(2)} \\ & + 2\bar{v}(n + m)[a_{n-1}^{(1)} - a_n^{(1)}] - \bar{v} a_{n-1}^{(1)} + m(1 - v)[b_{n-1}^{(1)} - b_n^{(1)}] \end{aligned} \tag{51}$$

$$\begin{aligned} \bar{E}_2^{(2)} = {} & \bar{E}_{21} a_{n-1}^{(2)} + \bar{E}_{22} b_{n-1}^{(2)} \\ & + m(1 - v)[a_{n-1}^{(1)} - a_n^{(1)}] - 2(n + m)[b_{n-1}^{(1)} - b_n^{(1)}] + b_{n-1}^{(1)} \end{aligned} \tag{52}$$

In a similar manner, we obtain the solutions for the system of differential equations (23) and (24) for the case (ii), i.e. shear modulus decreasing with depth. It can be shown that the corresponding power series coefficients for case (ii), designated by a star, are determined from those of case (i) by the simple formula

$$a_n^{*(i)} = \frac{a_n^{(i)}}{\Xi_0^* - 1} \tag{53}$$

$$b_n^{*(i)} = \frac{b_n^{(i)}}{\Xi_0^* - 1} \tag{54}$$

A direct consequence of this is the fact that the power series converge for $G_0 \geqslant G_\infty$ only when

$$\Xi_0^* < 0,5 \Leftrightarrow \Xi_0 > -1 \tag{55}$$

The solutions for the stresses are obtained by substituting the above solutions for the displacements into the stress-displacement relationships (4)–(7):

$$\sigma_{rr}(r, z) = \frac{2G_\infty(1-\xi)}{1-2v} \int_0^\infty k[(1-v)k\bar{u}(k, \xi) - v\alpha\xi\bar{w}'(k, \xi)] J_0(kr)\,\mathrm{d}k - \frac{2G_\infty(1-\xi)}{r} \int_0^\infty k\bar{u}(k, \xi) J_1(kr)\,\mathrm{d}k \tag{56}$$

$$\sigma_{\phi\phi}(r, z) = 2G_\infty(1-\xi)\frac{v}{1-2v} \int_0^\infty k[k\bar{u}(k, \xi) - \alpha\xi\bar{w}'(k, \xi)] J_0(kr)\,\mathrm{d}k + \frac{2G_\infty(1-\xi)}{r} \int_0^\infty k\bar{u}(k, \xi) J_1(kr)\,\mathrm{d}k \tag{57}$$

$$\sigma_{zz}(r, z) = \frac{2G_\infty(1-\xi)}{1-2v} \int_0^\infty k[vk\bar{u}(k, \xi) - \alpha(1-v)\xi\bar{w}'(k, \xi)] J_0(kr)\,\mathrm{d}k \tag{58}$$

$$\sigma_{rz}(r, z) = \sigma_{zr}(r, z) = -G_\infty(1-\xi) \int_0^\infty k[\alpha\xi\bar{u}'(k, \xi) + k\bar{w}(k, \xi)] J_1(kr)\,\mathrm{d}k \tag{59}$$

SOLUTION FOR A POINT LOAD

For the solution of the boundary value problem the point load is expressed by[20]

$$q(r) = \frac{Q}{2\pi} \int_0^\infty kJ_0(kr)\,\mathrm{d}k \tag{60}$$

and substituted together with the solution (58) for σ_{zz} into the boundary condition (8) yielding the following equation for $A_1(k)$ and $A_2(k)$:

$$A_1(k)RN_1(k) + A_2(k)RN_2(k) = -\frac{Q}{\pi}\frac{1-2v}{4G_0} \tag{61}$$

where

$$RN_i(k) = vk\bar{u}^{(i)}(k, \Xi_0) - \alpha(1-v)\Xi_0 \frac{\partial}{\partial\xi}\bar{w}^{(i)}(k, \Xi_0) \quad (i = 1, 2) \tag{62}$$

Similarly, inserting the expression (59) for σ_{rz} into boundary condition (9) yields

$$A_1(k)RT_1(k) + A_2(k)RT_2(k) = 0 \tag{63}$$

where

$$RT_i(k) = \alpha\Xi_0 \frac{\partial}{\partial\xi}\bar{u}^{(i)}(k, \Xi_0) + k\bar{w}^{(i)}(k, \Xi_0) \qquad (i = 1, 2) \tag{64}$$

Solving the system of linear equations (61) and (63) we obtain

$$A_1(k) = -\frac{Q}{2\pi G_0}\frac{RT_2(k)}{D(k)} \qquad A_2(k) = \frac{Q}{2\pi G_0}\frac{RT_1(k)}{D(k)} \tag{65}$$

where

$$D(k) = \frac{2}{1-2\nu}(RN_1(k)RT_2(k) - RN_2(k)RT_1(k)) \tag{66}$$

Substituting the expressions for $A_1(k)$ and $A_2(k)$ into the general solution (25)–(26) and performing the inverse Hankel transform results into the following expressions for the displacements at any point within the domain of the non-homogeneous elastic half-space:

$$u(r, z) = -\frac{Q}{2\pi G_0}\int_0^\infty k\frac{\Phi_u(k, \xi)}{D(k)}J_1(kr)\,\mathrm{d}k \tag{67}$$

$$w(r, z) = -\frac{Q}{2\pi G_0}\int_0^\infty k\frac{\Phi_w(k, \xi)}{D(k)}J_0(kr)\,\mathrm{d}k \tag{68}$$

where

$$\Phi_u(k, \xi) = RT_2(k)\bar{u}^{(1)}(k, \xi) - RT_1(k)\bar{u}^{(2)}(k, \xi) \tag{69}$$

$$\Phi_w(k, \xi) = RT_2(k)\bar{w}^{(1)}(k, \xi) - RT_1(k)\bar{w}^{(2)}(k, \xi) \tag{70}$$

The associated expressions for the stresses are obtained by substituting (25) and (26) into equations (56)–(59) and are omitted here for brevity. It can be easily shown that the stresses depend only on the non-homogeneity parameters Ξ_0 and α which describe the change of shear modulus with depth and not its absolute value. This behaviour is in accordance with that of the homogeneous case where the stresses do not depend on shear modulus.

For $\Xi_0 = 0$ expressions (67) and (68) reduce to those of Boussinesq at the surface of a homogeneous half-space.

$$u_H(r, 0) = -\frac{Q(1-2\nu)}{4\pi G_0}\int_0^\infty J_1(kr)\,\mathrm{d}k \tag{71}$$

$$w_H(r, 0) = \frac{Q(1-\nu)}{2\pi G_0}\int_0^\infty J_0(kr)\,\mathrm{d}k \tag{72}$$

For the numerical evaluation of the response in the interior of the homogeneous half-space we use equations (67) and (68) with a sufficiently small value $0 < \Xi_0 \ll 1$.

NUMERICAL RESULTS AND DISCUSSION

The evaluation of the improper integrals for the displacements and stresses requires some special treatment due to the presence of the oscillating and slowly decaying Bessel functions in the integrands. The slow decay of the Bessel function is mainly affecting the evaluation of the surface response, i.e. when the response is computed at the loading plane. To circumvent this difficulty we use both for stresses and displacements at the surface the following procedure which is described exemplarily for the vertical surface displacement.

The respective integral expression (68) is first written as

$$w(r,0)=\frac{Q}{G_0}\int_0^\infty F(k)J_0(kr)\,\mathrm{d}k \tag{73}$$

The integrand function $F(k)$ is bounded on the integration interval and takes the following values at its limits:

$$\text{for}\quad k\to 0\quad F(k)\to\frac{(1-\nu)}{2\pi}(1-\Xi_0)=F_0$$

$$\text{for}\quad k\to\infty\quad F(k)\to\frac{(1-\nu)}{2\pi}=F_\infty \tag{74}$$

The asymptotic value for large k corresponds to the solution of a homogeneous half-space with shear modulus G_0, whereas $k\to 0$ yields the solution for a half-space with shear modulus G_∞. The same behaviour is identified for case (ii), i.e. shear modulus decreasing with depth. Further, it can be easily shown that the independent variables k and α of the function F can be combined in a single dimensionless variable m, as defined by equation (30). The integrand function F is plotted in Figure 2 for representative values of the degree of non-homogeneity.

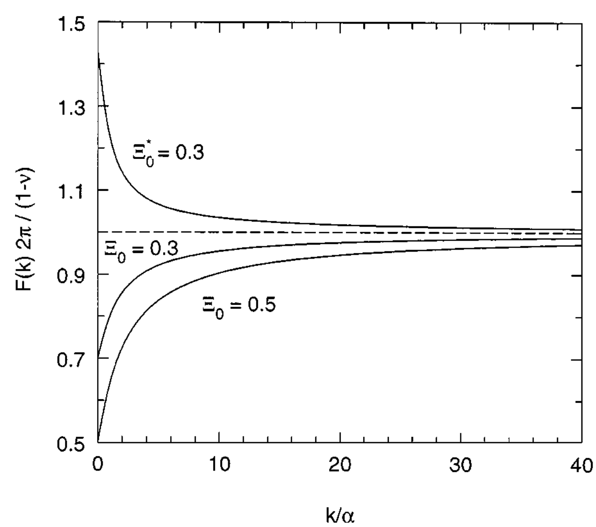

Figure 2. Integrand function of equation (73) for various values of the degree of non-homogeneity and for Poisson's ratio $\nu = 0{\cdot}3$

Based on the above property the integral (73) is written as

$$w(r,0)\frac{G_0}{Q}=\int_0^{\infty}[F(k)-F_{\infty}]J_0(kr)\,\mathrm{d}k+\int_0^{\infty}F_{\infty}J_0(kr)\,\mathrm{d}k$$
$$=\int_0^{\infty}[F(k)-F_{\infty}]J_0(kr)\,\mathrm{d}k+\frac{(1-\nu)}{2\pi}\frac{1}{r} \tag{75}$$

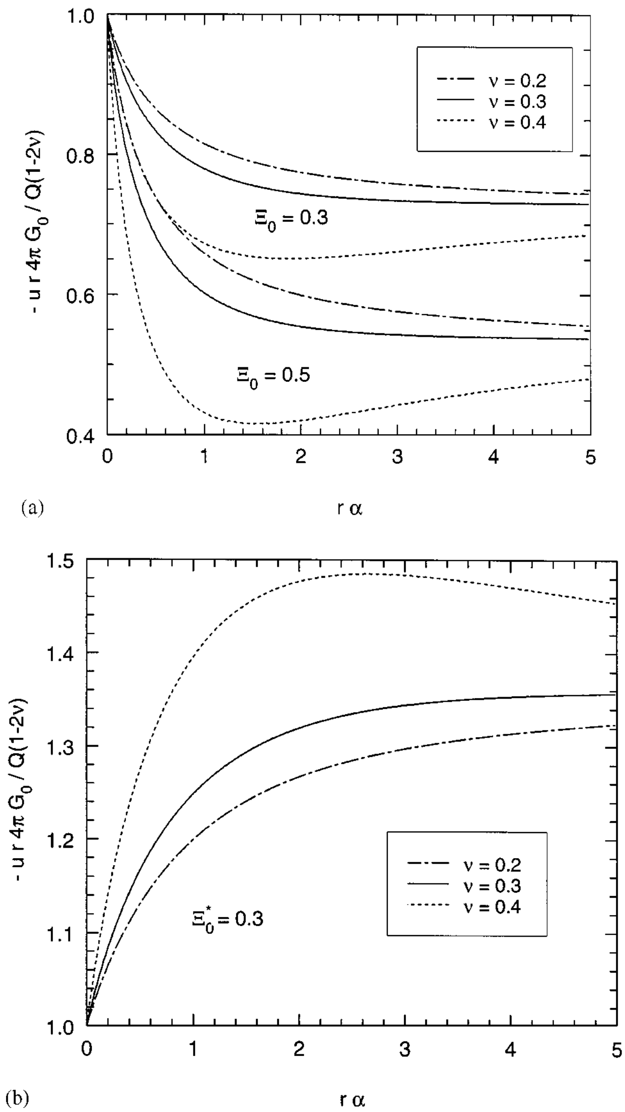

Figure 3. Horizontal surface displacement versus dimensionless distance for representative values of the degree of non-homogeneity and of the Poisson's ratio

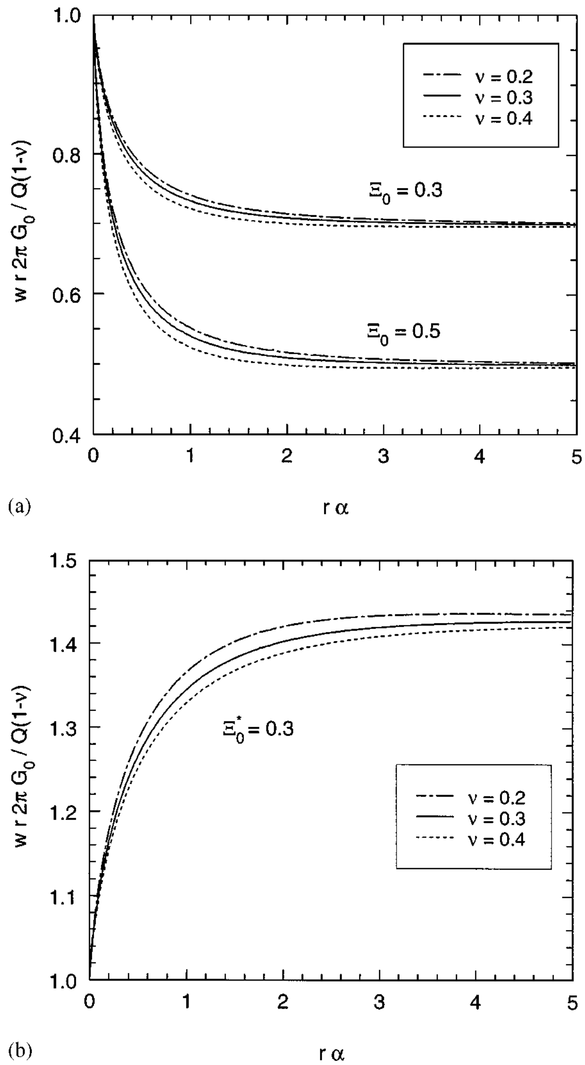

Figure 4. Vertical surface displacement versus dimensionless distance for representative values of the degree of non-homogeneity and of the Poisson's ratio

The fast decay of the integrand in (75) enables an accurate and efficient numerical integration using standard Gauss–Legendre quadrature and replacing the infinite upper limit of integration by a sufficiently large, finite one depending upon the value of Ξ_0.

In order to verify the derived solution, displacements and stresses in the interior of a homogeneous half-space were computed by setting either $\Xi_0 \to 0$ or $\alpha \to 0$ and compared with Boussinesq's solution showing excellent agreement.

We consider first the displacements at the surface of the nonhomogeneous half-space. It can be shown that the dependency on distance r and gradient of nonhomogeneity α is described by means of a single dimensionless variable $\bar{r} = r\alpha$. Figures 3 and 4 depict the variation of the

normalized horizontal and vertical displacement with $\bar{r}$ for selected values of the degree of non-homogeneity and of the Poisson's ratio. At the lower limit the curves converge to the solution for a homogeneous half-space with shear modulus G_0, whereas for large values $\bar{r}$ the curves asymptotically tend to the solution for a half-space with shear modulus G_∞. This provides an additional verification of the solution since for the former case $G \to G_0$ for $\alpha \to 0$ while for the latter case $G \to G_\infty$ for $\alpha \to \infty$. Further, it can be seen that the influence of Poisson's ratio on the vertical displacement can be represented in a good approximation by the factor $(1 - \nu)$ of the homogeneous case, whereas the dependency of the horizontal displacement on Poisson's ratio differs appreciably from that of the homogeneous case.

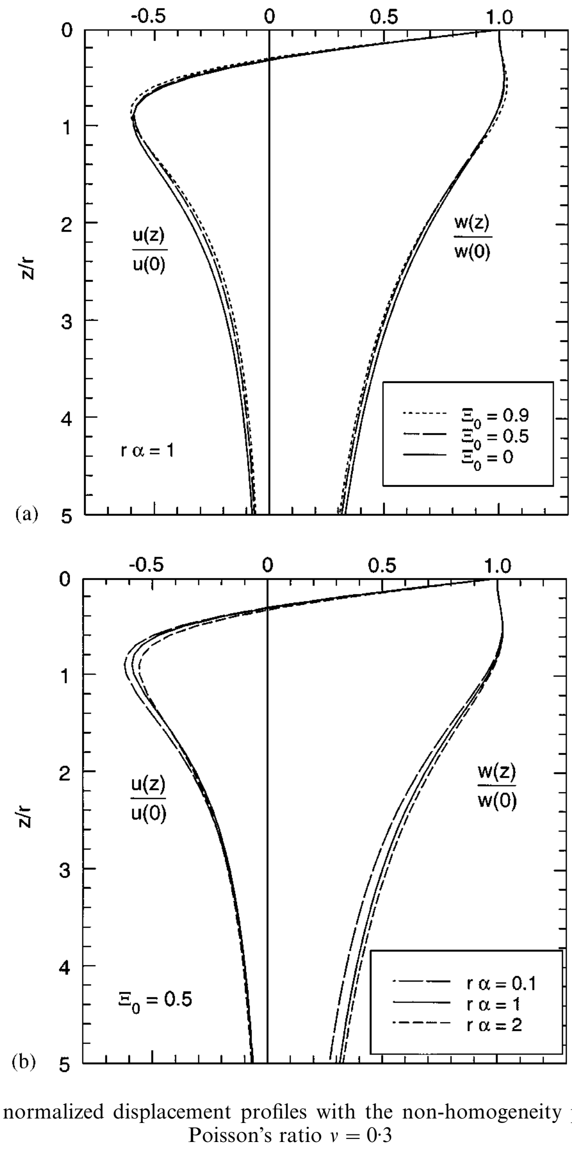

Figure 5. Variation of the normalized displacement profiles with the non-homogeneity parameters and distance for Poisson's ratio $\nu = 0{\cdot}3$

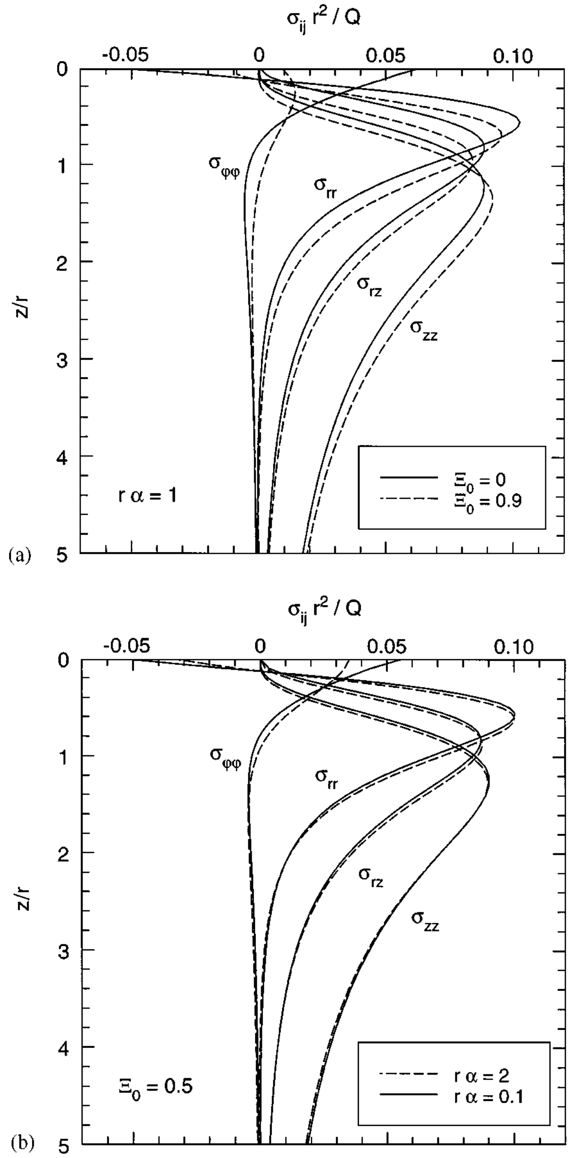

Figure 6. Dimensionless stress distribution with depth in dependence on the non-homogeneity parameters and distance for Poisson's ratio $\nu = 0{\cdot}3$

The distribution with depth of the displacements is shown in Figure 5 in dependency on the distance ratio z/r for various values of Ξ_0 and $\bar{r}$. The representation of normalized desplacements $u(z)/u(0)$ and $w(z)/w(0)$ in dependency on z/r yields for the homogeneous case a single curve for given Poisson's ratio. It can be seen that all depth profiles are very close to the one of the homogeneous case.

Next, we investigate the effects of non-homogeneity on the stress field at the surface. The compression positive sign convention is adopted. The non-vanishing radial and hoop stresses, σ_{rr} and $\sigma_{\phi\phi}$, respectively, exactly cancel and are related to the horizontal surface displacement by

$$\sigma_{rr}(r, 0) = -\,\sigma_{\phi\phi}(r, 0) = \frac{u(r, 0)G_0}{2r} \tag{76}$$

like in the homogeneous case. Thus, the dependency of σ_{rr} and $\sigma_{\phi\phi}$ on Poisson's ratio and degree of non-homogeneity may be identified directly from Figure 3.

The stress field within the half-space is depicted in dimensionless form in Figure 6 for given Poisson's ratio and for representative values of the non-homogeneity parameters revealing the rather insignificant dependency on the non-homogeneity gradient α, except for the stresses σ_{rr} and $\sigma_{\phi\phi}$ near the surface. The influence of Ξ_0 on the stress field, however, can not be neglected. Furthermore, it can be shown that unlike the homogeneous case the axial stress σ_{zz} and the shear stress σ_{rz} exhibit a dependency on Poisson's ratio which weakens with decreasing Ξ_0 and vanishes for $\Xi_0 = 0$.

CONCLUSIONS

Analytical solutions have been presented for the displacement and stress field in the interior of a continuously non-homogeneous elastic soil due to a vertical surface point load by using classical integral transform techniques and the extended power series method for deriving the solution in the transform domain. The function chosen for the depth variation of shear modulus is bounded at infinity and is capable of describing both increasing and decreasing stiffness with depth. The numerical results obtained demonstrate the pronounced effect of the non-homogeneity on the response at the surface, i.e. at the level of loading, whereas the normalized profiles of the displacements are almost unaffected by the non-homogeneity. The stress field in the interior shows a dependency on the degree of non-homogeneity while the influence of the gradient of non-homogeneity is confined to a near-surface region. In all cases non-homogeneity leads to an amplification of the influence of Poisson's ratio on displacements and stresses.

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